

A SPECTRAL SEQUENCE FOR FUSION SYSTEMS.

ANTONIO DÍAZ RAMOS

ABSTRACT. We build a spectral sequence converging to the cohomology of a fusion system. This spectral sequence is related to the Lyndon-Hochschild-Serre spectral sequence but does not always coincide with it. We give examples and prove Tate's p -nilpotency criterion following its original proof and using this new spectral sequence.

1. INTRODUCTION.

Let $K \trianglelefteq G$ be a normal subgroup of the finite group G and consider the extension

$$K \rightarrow G \rightarrow G/K.$$

The Lyndon-Hochschild-Serre spectral sequence of this short exact sequence is a important tool to analyze the cohomology of G with coefficients in the $\mathbb{Z}G$ -module M . It has second page $E_2^{p,q} = H^p(G/K; H^q(K; M))$ with G/K acting on $H^q(K; M)$ and converges to $H^{p+q}(G; M)$.

Our aim in this work is to construct a related spectral sequence in the ambient of fusion systems. This concept was originally introduced by Puig and developed by Broto, Levi and Oliver in [3], where we refer the reader for notation. It consists of a category \mathcal{F} with objects the subgroups of a finite p -subgroup S and morphisms bound by axioms that mimic properties of conjugation morphisms.

In the setup of fusion systems the concept of short exact sequence is an evasive one: Let \mathcal{F} be a fusion system over the p -group S . For a strongly \mathcal{F} -closed subgroup T of S there is a quotient fusion system \mathcal{F}/T [8, 5.10]. Nevertheless, in general there is no normal fusion subsystem of \mathcal{F} that would play the role of the kernel of the morphism of fusion systems $\mathcal{F} \rightarrow \mathcal{F}/T$ [2, Remark 8.7]. So the answer to [13, Conjecture 11] is negative and one cannot expect to construct a Lyndon-Hochschild-Serre spectral sequence for fusion systems. In any case, here we are able to construct a spectral sequence that converges to the cohomology of \mathcal{F} , $H^*(\mathcal{F}; M)$, where M is a $\mathbb{Z}_{(p)}$ -module with trivial action of S .

1.1. Theorem. *Let \mathcal{F} be a fusion system over the p -group S , T a strongly \mathcal{F} -closed subgroup of S and M a $\mathbb{Z}_{(p)}$ -module with trivial S -action. Then there is a first quadrant cohomological spectral sequence with second page*

$$E_2^{p,q} = H^p(S/T; H^q(T; M))^{\mathcal{F}}$$

and converging to $H^{p+q}(\mathcal{F}; M)$.

The notation \mathcal{F} will be fully described in Section 2 and must be thought as taking \mathcal{F} -stable elements: consider for each subgroup P of S the Lyndon-Hochschild-Serre

spectral sequence of the extension

$$P \cap T \rightarrow P \rightarrow P/P \cap T \cong PT/T$$

converging to $H^*(P; M)$. A morphism $\varphi \in \text{Hom}_{\mathcal{F}}(P, Q)$ induces a morphism φ^* between the spectral sequences corresponding to Q and P . Hence we have a contravariant functor from \mathcal{F} to spectral sequences. The inverse limit spectral sequence or spectral sequence of \mathcal{F} -stable elements has entry $E_2^{p,q}$ equal to $H^p(S/T; H^q(T; M))^{\mathcal{F}}$, i.e., the elements z from

$$H^p(S/T; H^q(T; M))$$

such that $\varphi^*(z) = \text{res}(z)$, where $\varphi \in \text{Hom}_{\mathcal{F}}(P, S)$ and $\text{res} = \iota^*$ is restriction in cohomology for the inclusion $P \xrightarrow{\iota} S$. Hence $H^*(S/T; H^*(T; M))^{\mathcal{F}}$ is a subalgebra of the differential graded algebra $H^*(S/T; H^*(T; M))$ and its differential is just restriction of the differential of the latter. This is useful in computations. The theorem states that abutment of this spectral sequence is $H^*(\mathcal{F}; M)$.

When restricted to the fusion system $\mathcal{F}_S(G)$ of a finite group G with Sylow p -subgroup S we obtain a spectral sequence substantially distinct from the Lyndon-Hochschild-Serre spectral sequence but still converging to $H^*(G; M)$. Here M is a $\mathbb{Z}_{(p)}$ -module with trivial G -action (and hence trivial S -action) and so $H^*(G; M) = H^*(\mathcal{F}; M)$. The strongly $\mathcal{F}_S(G)$ -closed subgroup to be considered is $T = K \cap S$. We give examples (Section 4) that show that these spectral sequences might have different E_2 -pages.

As an application of the spectral sequence in Theorem 1.1 we prove Tate's criterion for p -nilpotency of finite groups translated to the setup of fusion systems:

1.2. Corollary ([14]). *Let \mathcal{F} be a fusion system over the p -group S . If the restriction map $H^1(\mathcal{F}; \mathbb{F}_p) \rightarrow H^1(S; \mathbb{F}_p)$ is an isomorphism then $\mathcal{F} = \mathcal{F}_S(S)$.*

This result was already proven in [9] using transfer for fusion systems and in [5] by topological methods. Here we are able to mimic Tate's cohomological original proof that relies on the five terms exact sequence associated to the Lyndon-Hochschild-Serre spectral sequence. We use instead the spectral sequence of 1.1, showing that this new spectral sequence fits in the p -local setup of fusion systems.

1.3. Remark. Let \mathcal{F} be a fusion system over the p -group S . An \mathcal{F} -stable $\mathbb{Z}_{(p)}S$ -module is an $\mathbb{Z}_{(p)}S$ -module such that for every morphism $\varphi : P \rightarrow Q$ in \mathcal{F} we have that

$$\varphi(p) \cdot m = p \cdot m$$

for every $p \in P$ and $m \in M$. Any $\mathbb{Z}_{(p)}$ -module M is an \mathcal{F} -stable $\mathbb{Z}_{(p)}S$ -module with the trivial action of S . Theorem 1.1 holds with M an \mathcal{F} -stable $\mathbb{Z}_{(p)}S$ -module instead of the trivial $\mathbb{Z}_{(p)}$ -module M .

Organization of the paper: In Section 2 \mathcal{F} -stable elements are defined and some related results introduced. In Section 3 the spectral sequence is built and Theorem A is proven as Theorem 3.3. In Section 4 we compare the spectral sequence from Theorem 1.1 to the Lyndon-Hochschild-Serre spectral and give some examples. In Section 5 we prove Tate's theorem.

Acknowledgments: I would to thank A. Viruel for fruitful conversations and in particular for the idea of using a finite group model to construct the functorial resolutions in Lemma 3.2.

2. COHOMOLOGY AND \mathcal{F} -STABLE ELEMENTS.

Throughout this section \mathcal{F} denotes a fusion system over the p -group S . We start introducing some notation: if $A : \mathcal{F} \rightarrow \mathcal{C}$ is a contravariant functor and \mathcal{C} is any category then by φ^* we denote the value $A(\varphi)$ for φ a morphism in \mathcal{F} . For $\varphi = \iota$, the inclusion of P into S , we write $res := \iota^*$. If \mathcal{C} is a complete category then we denote by $A^{\mathcal{F}}$ the inverse limit over \mathcal{F} of this functor:

$$A^{\mathcal{F}} := \varprojlim_{\mathcal{F}} A.$$

If there is a functor $U : \mathcal{C} \rightarrow \mathbf{Sets}$ that creates (inverse) limits then there is nice description of $A^{\mathcal{F}}$:

2.1. Lemma. *Let $A : \mathcal{F} \rightarrow \mathcal{C}$ be a contravariant functor with \mathcal{C} complete and such that $U : \mathcal{C} \rightarrow \mathbf{Sets}$ creates limits. Then*

$$A^{\mathcal{F}} = A(S)^{\mathcal{F}} := \{z \in A(S) \mid res(z) = \varphi^*(z) \text{ for each } \varphi \in \text{Hom}_{\mathcal{F}}(P, S)\} \subseteq A(S).$$

We call the elements in $A(S)^{\mathcal{F}}$ the \mathcal{F} -stable elements in $A(S)$. The category $\mathcal{C} = \mathbf{Ab}$ of abelian groups is complete and the forgetful functor $U : \mathcal{C} \rightarrow \mathbf{Sets}$ creates limits. Hence the lemma applies. For the complete category $\mathcal{C} = \text{CCh}(\mathbf{Ab})$ of cochain complexes limits are constructed dimension-wise and the result of the lemma also applies, i.e., for any functor $A : \mathcal{F} \rightarrow \text{CCh}(\mathbf{Ab})$ we have that

$$A^{\mathcal{F}} = A(S)^{\mathcal{F}}.$$

For such a functor we can consider the cohomology $H^*(A^{\mathcal{F}}) = H^*(A(S)^{\mathcal{F}})$ of $A(S)^{\mathcal{F}} \in \text{CCh}(\mathbf{Ab})$. Notice that we also have functors $H^n(A) : \mathcal{F} \rightarrow \mathbf{Ab}$ obtained by taking cohomology at degree n . Hence we can also consider the inverse limits $H^*(A)^{\mathcal{F}} = H^*(A(S))^{\mathcal{F}}$. We are interested in functors A for which taking \mathcal{F} -stable elements and cohomology commute. It turns out that having transfers available and working on $\mathbb{Z}_{(p)}$ -modules is sufficient. The following definition involves a composition being equal to multiplication by some index. Notice that this is not part of the actual definition of Mackey functor, for which we refer the reader to [15], but it is satisfied by the transfer in cohomology of groups.

2.2. Definition. A contravariant functor $A : \mathcal{F} \rightarrow \text{CCh}(\mathbf{Ab})$ is a Mackey functor if there exists a covariant functor $B : \mathcal{F} \rightarrow \text{CCh}(\mathbf{Ab})$ with $A(P) = B(P)$ for every $P \leq S$ and for the inclusion $\iota : P \rightarrow S$ the map $B(\iota) \circ A(\iota) : A(S) \rightarrow A(S)$ is multiplication by $|S : P|$.

2.3. Remark. The map $B(\iota_P^Q)$ for the inclusion $\iota_P^Q : P \rightarrow Q$ is called the *transfer* from P into Q for good reasons (see Example 2.6). Given the contravariant functor A , the transfers are all what is needed to define the covariant part B : for any morphism $\varphi : P \rightarrow Q$ define $B(\varphi) : A(P) \rightarrow A(Q)$ by $B(\varphi) = B(\iota_{\varphi(P)}^Q) \circ A(\varphi^{-1})$. Such a B becomes functorial if for any $P \leq Q \leq R$ we have $B(\iota_Q^R) \circ B(\iota_P^Q) = B(\iota_P^R)$ and for any $P \leq Q \xrightarrow{\varphi} S$ we have $A(\varphi^{-1}) \circ B(\iota_P^Q) = B(\iota_{\varphi(P)}^{(Q)}) \circ A((\varphi|_P)^{-1})$.

We also note that in [12] Ragnarsson and Stancu give a definition of Mackey functor in the setup of fusion systems and characterize the \mathcal{F} -stable elements for such a functor as the fixed points of the idempotent associated to \mathcal{F} .

2.4. Lemma. *Let \mathcal{F} be a fusion system over the p -group S and $A : \mathcal{F} \rightarrow \text{CCh}(\mathbb{Z}_{(p)})$ a Mackey functor. Then*

$$H^*(A(S)^\mathcal{F}) \cong H^*(A(S))^\mathcal{F}.$$

Proof. For any $n \geq 0$ write $A^n : \mathcal{F} \rightarrow \mathbb{Z}_{(p)} - \text{mod}$ for the degree n component of A . We have subfunctors $\text{Ker } A^n$ and $\text{Im } A^n$ of A^n defined as the kernel and image of the differential on dimension n . We have the chain of isomorphisms:

$$H^n(A(S)^\mathcal{F}) \stackrel{(1)}{\cong} \left(\frac{\text{Ker } A^n}{\text{Im } A^n} \right)^\mathcal{F} \stackrel{(2)}{\cong} \frac{(\text{Ker } A^n)^\mathcal{F}}{(\text{Im } A^n)^\mathcal{F}} \stackrel{(3)}{\cong} \frac{\text{Ker } A^n(S)^\mathcal{F}}{\text{Im } A^n(S)^\mathcal{F}} \stackrel{(4)}{\cong} H^n(A(S)^\mathcal{F}),$$

where $\text{Ker } A^n(S)^\mathcal{F}$ and $\text{Im } A^n(S)^\mathcal{F}$ denote respectively the kernel and image in $A(S)^\mathcal{F}$ of the (restriction of the) differentials $A^{n-1}(S) \xrightarrow{d^{n-1}} A^n(S) \xrightarrow{d^n} A^{n+1}(S)$.

Here the isomorphisms (1) and (4) are just the definition of the left hand side and right hand side terms respectively. Recall that by Lemma 2.1 $(\text{Ker } A^n)^\mathcal{F} = (\text{Ker } A^n(S))^\mathcal{F}$ and $(\text{Im } A^n)^\mathcal{F} = (\text{Im } A^n(S))^\mathcal{F}$. The isomorphism (3) is consequence of the following two readily checked equalities:

$$(\text{Ker } A^n(S))^\mathcal{F} = \text{Ker } A^n(S)^\mathcal{F} = A^n(S)^\mathcal{F} \cap \text{Ker } d^n$$

and

$$(\text{Im } A^n(S))^\mathcal{F} = \text{Im } A^n(S)^\mathcal{F} = A^n(S)^\mathcal{F} \cap \text{Im } d^{n-1}.$$

For the isomorphism (2) we want to see that taking inverse limit commutes with taking quotient for the pair of given functors. There is a map

$$(2.5) \quad (\text{Ker } A^n(S))^\mathcal{F} \rightarrow \left(\frac{\text{Ker } A^n(S)}{\text{Im } A^n(S)} \right)^\mathcal{F} = H^n(A(S))^\mathcal{F}$$

which is the restriction of the map

$$\pi : \text{Ker } d^n \twoheadrightarrow H^n(A(S))$$

to $(\text{Ker } A^n(S))^\mathcal{F} = A^n(S)^\mathcal{F} \cap \text{Ker } d^n \leq \text{Ker } d^n$. The kernel of (2.5) is given by

$$(\text{Ker } A^n(S))^\mathcal{F} \cap \text{Im } d^{n-1} = A^n(S)^\mathcal{F} \cap \text{Ker } d^n \cap \text{Im } d^{n-1} = \text{Im } A^n(S)^\mathcal{F},$$

as $\text{Im } d^{n-1} \leq \text{Ker } d^n$ and using the equalities above.

So to finish the proof it only remains left to prove that the map (2.5) is surjective. Consider an (S, S) -biset Ω satisfying the properties listed in [3, Proposition 5.5]. Write the biset Ω as

$$\Omega = \coprod S \times \varphi S,$$

where the disjoint union runs over the set of morphisms of \mathcal{F} with possible repetitions. For any such morphism $\varphi \in \text{Hom}_\mathcal{F}(P, S)$ we have the composition

$$A(S) \xrightarrow{A(\varphi)} A(P) = B(P) \xrightarrow{B(\iota)} B(S) = A(S),$$

where $\iota : P \rightarrow S$ is the inclusion. Then we can define a map $A(\Omega) : A(S) \rightarrow A(S)$ by

$$A(\Omega) := \sum B(\iota) \circ A(\varphi).$$

We claim that $A(S)^\mathcal{F} = \text{Im } A(\Omega)$. That $\text{Im } A(\Omega) \leq A(S)^\mathcal{F}$ is a direct consequence of property [3, Proposition 5.5(b)]. Consider now $a \in A(S)^\mathcal{F}$. For any morphism $\varphi \in \text{Hom}_\mathcal{F}(P, S)$ we have that

$$(B(\iota) \circ A(\varphi))(a) = B(\iota)(A(\varphi)(a)) = B(\iota)(A(\iota)(a)) = |S : P| \cdot a = \frac{|S \times_\varphi S|}{|S|} \cdot a$$

by Definition 2.2. Hence $A(\Omega)(a) = \frac{|\Omega|}{|S|} \cdot a$ if $a \in A(S)^{\mathcal{F}}$. Because of property [3, Proposition 5.5(c)] this number $z = \frac{|\Omega|}{|S|}$ is invertible mod p and hence $A(\Omega)(a/z) = a$ and $A(S)^{\mathcal{F}} \leq \text{Im } A(\Omega)$.

Because A and B are functors that land in $\text{CCh}(\mathbb{Z}_{(p)})$ they induce for each degree n (sub)functors $\text{Ker } A^n, \text{Ker } B^n : \mathcal{F} \rightarrow \mathbb{Z}_{(p)} - \text{mod}$ as well as functors $H^n(A), H^n(B) : \mathcal{F} \rightarrow \mathbb{Z}_{(p)} - \text{mod}$. Because multiplication is preserved under restriction and passing to cohomology these functors become Mackey functors and the same argument above shows then that

$$H^n(A(S))^{\mathcal{F}} = \text{Im } H^n(A)(\Omega)$$

and

$$(\text{Ker } A^n)^{\mathcal{F}} = \text{Im}(\text{Ker } A^n)(\Omega).$$

Now we can finish the proof the surjectivity of (2.5). First notice that the following diagram commutes:

$$\begin{array}{ccc} \text{Ker } d^n & \xrightarrow{\pi} & H^n(A(S)) \\ \downarrow A(\Omega)|_{\text{Ker } d^n} & & \downarrow H^n(A)(\Omega) \\ \text{Ker } d^n & \xrightarrow{\pi} & H^n(A(S)). \end{array}$$

We know that $H^n(A(S))^{\mathcal{F}} = \text{Im } H^n(A)(\Omega) = H^n(A)(\Omega)(H^n(A(S)))$ and this equals

$$(H^n(A)(\Omega) \circ \pi)(\text{Ker } d^n) = (\pi \circ A(\Omega))(\text{Ker } d^n) = \pi((\text{Ker } A^n)^{\mathcal{F}}) = \pi((\text{Ker } A^n(S))^{\mathcal{F}}).$$

□

2.6. Example. For M a p -local abelian group, i.e., a $\mathbb{Z}_{(p)}$ -module, in [3] the cohomology $H^*(\mathcal{F}; M)$ is defined as $H^*(S; M)^{\mathcal{F}}$ where $H^*(\cdot; M) : \mathcal{F} \rightarrow \mathbb{Z}_{(p)} - \text{mod}$ is the functor that takes $P \leq S$ to its cohomology with (trivial) M coefficients $H^*(P; M)$. This is Mackey functor according to Definition (2.2) using as covariant part the transfer in cohomology. A closer analysis shows that the transfer can be defined at the level of cochains in such a way that restriction followed by transfer is still multiplication by the index. (See next section.) So if we choose functorial cochain complexes $C^*(P; M)$ equipped with transfer at the level of cochains the preceding lemma shows that

$$H^*(\mathcal{F}; M) = H^*(S; M)^{\mathcal{F}} = H^*(C^*(S; M))^{\mathcal{F}}.$$

3. CONSTRUCTION OF THE SPECTRAL SEQUENCE.

Given a ring R , one way of constructing the Lyndon-Hochschild-Serre spectral sequence for a short exact sequence of groups

$$0 \rightarrow K \rightarrow G \rightarrow G/K \rightarrow 0$$

and an RG -module M is to use the spectral sequence associated to the double complex $A^{p,q} = \text{Hom}_{RG/K}(B_{G/K}^p, \text{Hom}_{RK}(B_G^q, M))$, where $B_{G/K}^*$ is a projective $R(G/K)$ -resolution of R and B_G^* is a projective RG -resolution of R (restricted to RK) [10]. This spectral sequence satisfies

$$E_2^{p,q} \cong H^p(G/K; H^q(K; M)) \Rightarrow H^{p+q}(G; M).$$

On the other hand, by the classical result [6, XII.10.1], attributed to Tate by Cartan and Eilenberg, for $R = \mathbb{Z}_{(p)}$ we know that $H^*(G; M)$ consists exactly of the G -stable elements in $H^*(S; M)$, where S is a Sylow p -subgroup of G . Using the notation of section 2 we can rephrase this as

$$H^*(G; M) = H^*(S; M)^{\mathcal{F}},$$

where $\mathcal{F} = \mathcal{F}_S(G)$ is the fusion system associated to G .

In this section we answer the question of what spectral sequence do we obtain if we consider instead the the double complex $(A^{*,*})^{\mathcal{F}}$ of \mathcal{F} -stable elements in the double complex

$$A^{p,q} = \text{Hom}_{\mathbb{Z}_{(p)}S/T}(B_{S/T}^p, \text{Hom}_{\mathbb{Z}_{(p)}T}(B_S^q, M)).$$

Here $A^{*,*}$ is the double complex associated to the short exact sequence of Sylow p -subgroups

$$0 \rightarrow T \rightarrow S \rightarrow S/T \rightarrow 0$$

and $T = K \cap S$ is a Sylow p -subgroup of K .

We face directly the generic fusion system case: let \mathcal{F} be a fusion system over the p -group S , T a strongly \mathcal{F} -closed subgroup of S and M a $\mathbb{Z}_{(p)}$ -module with trivial S -action. Notice that from the finite group case situation $K \trianglelefteq G$ we go to the fusion system case by considering $\mathcal{F} = \mathcal{F}_S(G)$ and $T = K \cap S$. (See Section 4.)

The first task to accomplish if we wish to consider \mathcal{F} -stable elements in the double complex of the Sylow p -subgroups is to convert this double complex into a contravariant functor $A^{*,*}$ from \mathcal{F} into double complexes of $\mathbb{Z}_{(p)}$ -modules $CCh^2(\mathbb{Z}_{(p)})$. Choose projective resolutions B_P^* and $B_{PT/T}^*$ of the trivial module $\mathbb{Z}_{(p)}$ over $\mathbb{Z}_{(p)}P$ and $\mathbb{Z}_{(p)}PT/T$ respectively for each p -subgroup $P \leq S$ in a (covariant) functorial way, for example, using the bar resolution for groups. In the lemma below we show concrete resolutions that satisfy additional properties.

If P is a subgroup of S we define $A^{*,*}(P)$ as the double complex associated to the short exact sequence

$$0 \rightarrow P \cap T \rightarrow P \rightarrow P/P \cap T \cong PT/T \rightarrow 0.$$

More precisely, we define

$$A^{p,q}(P) = \text{Hom}_{\mathbb{Z}_{(p)}PT/T}(B_{PT/T}^p, \text{Hom}_{\mathbb{Z}_{(p)}P \cap T}(B_P^q, M)),$$

where the action of $PT/T \cong P/P \cap T$ on $\text{Hom}_{\mathbb{Z}_{(p)}P \cap T}(B_P^q, M)$ is given by

$$\bar{p} \cdot f(x) = p \cdot f(p^{-1} \cdot x),$$

where $p \in P$, $x \in B_P^q$, f is a cochain and the action is well defined because f is a $\mathbb{Z}_{(p)}P \cap T$ -map.

To define A on morphisms notice that any morphism $\varphi \in \text{Hom}_{\mathcal{F}}(P, Q)$ takes $P \cap T$ to $Q \cap T$ as T is strongly \mathcal{F} -closed. Hence it induces a homomorphism $\bar{\varphi} : PT/T \rightarrow QT/T$ via the isomorphisms $PT/T \cong P/P \cap T$ and $QT/T \cong Q/Q \cap T$.

3.1. Remark. Notice that according to [8, 5.10] the morphism $\bar{\varphi}$ belongs to \mathcal{F}/T , i.e., there exists $\psi \in \text{Hom}_{\mathcal{F}}(PT, QT)$ such that the induced map $\bar{\psi} : PT/T \rightarrow QT/T$ coincides with $\bar{\varphi}$.

In the next lemma we define how does A act on morphisms and also we show that there is a covariant part B that makes A close to a Mackey functor (Definiton 2.2):

3.2. Lemma. *Let \mathcal{F} be a fusion system over the p -subgroup S and let M be a $\mathbb{Z}_{(p)}$ -module with trivial S -action. Let T be a \mathcal{F} -strongly closed subgroup of S . Then there exist functors A (contravariant) and B (covariant) from \mathcal{F} to double cochain complexes of $\mathbb{Z}_{(p)}$ -modules ($\text{CCh}^2(\mathbb{Z}_{(p)})$) with value on $P \leq S$ given by*

$$A^{*,*}(P) = B^{*,*}(P) = \text{Hom}_{\mathbb{Z}_{(p)}PT/T}(B_{PT/T}^*, \text{Hom}_{\mathbb{Z}_{(p)}P \cap T}(B_P^*, M)),$$

where B_P^* and $B_{PT/T}^*$ are projective resolutions of $\mathbb{Z}_{(p)}$ over $\mathbb{Z}_{(p)}P$ and $\mathbb{Z}_{(p)}PT/T$ respectively and such that for the inclusion $\iota : P \rightarrow Q$ we have that

$$B(\iota) \circ A(\iota) : A(Q) \rightarrow A(Q)$$

is multiplication by $|Q : P|^2$.

Proof. By [11, Theorem 1] there exists a finite group G with $S \leq G$ (S not necessarily a Sylow p -subgroup of G) such that $\mathcal{F} = \mathcal{F}_S(G)$ and a finite group G' with $S/T \leq G'$ and $\mathcal{F}/T = \mathcal{F}_{S/T}(G')$ (S/T not necessarily a Sylow p -subgroup of G'). Consider free resolutions F^* and F'^* of the trivial module $\mathbb{Z}_{(p)}$ over $\mathbb{Z}_{(p)}G$ and $\mathbb{Z}_{(p)}G'$ respectively. The restriction B_P^* ($B_{PT/T}^*$) of F^* (F'^*) to a subgroup P of S (PT/T of S/T) is a free resolution of the trivial module $\mathbb{Z}_{(p)}$ over $\mathbb{Z}_{(p)}P$ ($\mathbb{Z}_{(p)}PT/T$).

Each morphism $\varphi : P \rightarrow Q$ in \mathcal{F} is induced by conjugation by an element in G and hence induces a morphism $\varphi_* : B_P^* \rightarrow B_Q^*$ that satisfies $\varphi_*(p \cdot b) = \varphi(p) \cdot \varphi_*(b)$ for any $p \in P$ and $b \in B_P^*$. Analogously, each morphism $\bar{\varphi} : PT/T \rightarrow QT/T$ in \mathcal{F}/T is induced by conjugation by an element in G' and hence induces a morphism $\bar{\varphi}_* : B_{PT/T}^* \rightarrow B_{QT/T}^*$ satisfying $\bar{\varphi}_*(\bar{p} \cdot b) = \bar{\varphi}(\bar{p}) \cdot \bar{\varphi}_*(b)$ for any $\bar{p} \in PT/T$ and $b \in B_{PT/T}^*$.

For any morphism $\varphi : P \rightarrow Q$ we define $A(\varphi) : A(Q) \rightarrow A(P)$ as the map taking the cochain f to the cochain $A(\varphi)(f)$ that on $x \in B_{PT/T}^*$ and $y \in B_P^*$ takes the value $f(\bar{\varphi}(x), \varphi(y))$. Notice that by Remark 3.1 the morphism $\bar{\varphi}$ belongs to \mathcal{F}/T . Because M has the trivial S -action it is readily checked that $A(\varphi)(f) \in A(P)$.

Let $V \leq W$ be two groups and N a $\mathbb{Z}_{(p)}W$ -module. Then we have a transfer map at the level of invariants $tr : N^V \rightarrow N^W$ given by

$$tr(n) = \sum_{w \in W/V} w \cdot n,$$

where w runs over coset representatives.

Now let ι be the inclusion between subgroups $P \leq Q$ of S and set N equal to the $\mathbb{Z}_{(p)}S$ -module $N := \text{Hom}_{\mathbb{Z}_{(p)}}(F^*, M)$. Then we have a transfer map

$$tr : \text{Hom}_{\mathbb{Z}_{(p)}P \cap T}(B_P^*, M) = N^{P \cap T} \rightarrow \text{Hom}_{\mathbb{Z}_{(p)}Q \cap T}(B_Q^*, M) = N^{Q \cap T}$$

given as above for the $\mathbb{Z}_{(p)}Q \cap T$ -module N and the inclusions $P \cap T \leq Q \cap T$. Setting now N' equal to the $\mathbb{Z}_{(p)}QT/T$ module

$$N' := \text{Hom}_{\mathbb{Z}_{(p)}}(F'^*, \text{Hom}_{\mathbb{Z}_{(p)}Q \cap T}(B_Q^*, M))$$

we get a transfer map tr' from

$$N'^{PT/T} = \text{Hom}_{\mathbb{Z}_{(p)}}(F'^*, \text{Hom}_{\mathbb{Z}_{(p)}Q \cap T}(B_Q^*, M))^{PT/T}$$

to

$$N'^{QT/T} = \text{Hom}_{\mathbb{Z}_{(p)}}(F'^*, \text{Hom}_{\mathbb{Z}_{(p)}Q \cap T}(B_Q^*, M))^{QT/T} = B(Q)$$

using the inclusion $PT/T \leq QT/T$. Finally we define $B(\iota)$ by taking the cochain f in $B(P)$ to the cochain $tr'(tr \circ f)$. Clearly $tr \circ f$ belongs to N' and to see that $B(\iota)$ is well defined one carefully checks that actually $tr \circ f \in N'^{PT/T}$.

To define $B(\psi)$ on a generic morphism $\psi : P \rightarrow Q$ in \mathcal{F} write $\psi = \iota \circ \varphi$ where $\varphi : P \rightarrow \psi(P)$ is an isomorphism and ι is the inclusion $\psi(P) \leq Q$ and set $B(\psi) = B(\iota) \circ A(\varphi^{-1})$. It is straightforward that A is a functor. Checking that B is a functor is a more lengthy computation that can be shortened using Remark 2.3. If $f \in A(Q)$, $P \leq Q$ and ι is the inclusion then an short computation yields that

$$(B(\iota) \circ A(\iota))(f) = \frac{|Q \cap T|}{|P \cap T|} \frac{|QT||T|}{|PT||T|} f = |Q : P| f.$$

□

3.3. Theorem. *Let \mathcal{F} be a fusion system over the p -group S , T a strongly \mathcal{F} -closed subgroup of S and M a $\mathbb{Z}_{(p)}$ -module with trivial S -action. Then there is a first quadrant cohomological spectral sequence with second page*

$$E_2^{p,q} = H^p(S/T; H^q(T; M))^{\mathcal{F}}$$

and converging to $H^{p+q}(\mathcal{F}; M)$.

3.4. Remark. As noted in the introduction each morphism $\varphi : P \rightarrow Q$ induces a morphism from the E_2 -page of the Lyndon-Hochschild-Serre spectral sequence of the short exact sequence

$$Q \cap T \rightarrow Q \rightarrow Q/Q \cap T \cong QT/T$$

to that of

$$P \cap T \rightarrow P \rightarrow P/P \cap T \cong PT/T.$$

This a morphism of differential graded algebras and hence the E_2 -page of the spectral sequence in the statement of the theorem is a differential graded subalgebra of $H^p(S/T; H^q(T; M))$. In particular, the differential in the E_2 -page of the spectral sequence of the theorem is the restriction to this subalgebra of the differential in the E_2 -page of the Lyndon-Hochschild-Serre spectral sequence for $T \rightarrow S \rightarrow S/T$.

Proof. We follow the construction of the Lyndon-Hochschild-Serre spectral sequence in MacLane's book [10, XI.10.1] via double complexes.

Our double complex is the double complex $A^{*,*}(S)^{\mathcal{F}}$ of \mathcal{F} -stable elements in the double complex $A^{*,*}(S)$ of the short exact sequence $T \rightarrow S \rightarrow S/T$. The \mathcal{F} -stability is with respect to the functor from \mathcal{F} to double complexes defined in Lemma 3.2.

There are two filtrations of the total complex $\text{Tot}(A^{*,*}(S)^{\mathcal{F}})$, namely, filtration by rows or filtration by columns. Both filtrations yield spectral sequences E' and E'' converging to $H^*(\text{Tot}(A^{*,*}(S)^{\mathcal{F}}))$. As usual, we will see that E'' collapses at the horizontal axis of the E_2'' -page as $H^*(\mathcal{F}; M)$ and that the page E_2' is precisely $H^p(S/T; H^q(T; M))^{\mathcal{F}}$.

The term $E_2''^{p,q}$ is exactly $H_v^q H_h^p(A^{*,*}(S)^{\mathcal{F}})$, where v and h mean that we use the vertical or horizontal differential respectively. The cohomology $H_h^p(A^{*,*}(S)^{\mathcal{F}})$ is exactly the cohomology of the complex

$$A^{*,q}(S)^{\mathcal{F}} = \text{Hom}_{\mathbb{Z}_{(p)} S/T}(B_{S/T}^*, \text{Hom}_{\mathbb{Z}_{(p)} T}(B_S^q, M))^{\mathcal{F}}.$$

Notice that $A^{*,q}(\cdot)$ is exactly the composition \tilde{A} of the contravariant functor A in Lemma 3.2 with the forgetful functor that forgets everything but the cochain

complex at $*, q$. Postcomposing B from Lemma 3.2 in exactly the same way we get a covariant functor \tilde{B} such that for the inclusion $\iota : P \rightarrow S$ we have that $\tilde{B}(\iota) \circ \tilde{A}(\iota)$ is multiplication by $|S : P|^2$. So \tilde{A} is not Mackey according to Definition 2.2 and we cannot apply Lemma 2.4 directly.

Nevertheless, the only point where the proof of the lemma does not work is that for the (S, S) -biset Ω we have that $\tilde{A}(\Omega)$ is not multiplication by $|\Omega|/|S|$ but by $\sum |S : P|^2$, where the sum runs over the morphisms $\varphi : P \rightarrow S$ involved in Ω . So it is enough to check that this sum $\sum |S : P|^2$ is invertible mod p . This is straightforward: write

$$\Omega = \coprod S \times \varphi S.$$

Then we know that $|\Omega|/|S|$ is invertible mod p . But $|\Omega| = \sum |S|^2/|P| = |S|k + |S|pq$, where k is the number of morphism $\varphi : S \rightarrow S$ appearing in the decomposition of Ω and q is an integer. Hence $|\Omega|/|S| = k \pmod{p}$ is invertible mod p . On the other hand,

$$\sum |S : P|^2 = \sum |S|^2/|P|^2 = k + pr$$

for some integer r and we are done.

So the conclusion of Lemma 2.4 holds for \tilde{A} and $H_h^p(A^{*,*}(S)^{\mathcal{F}}) = H_h^p(A^{*,*}(S))^{\mathcal{F}}$. Because multiplication by a scalar induces multiplication by the same scalar in cohomology the same argument shows that $H_v^q H_h^p(A^{*,*}(S)^{\mathcal{F}}) = H_v^q H_h^p(A^{*,*}(S))^{\mathcal{F}} = (H_v^q H_h^p(A^{*,*}(S)))^{\mathcal{F}}$. By [10, p.352] we know that $H_v^q H_h^p(A^{*,*}(S))$ is $H^p(S; M)$ if $q = 0$ and is zero if $q > 0$. Hence, E'' collapses at E_2'' and converges to $H^*(S; M)^{\mathcal{F}} = H^*(\mathcal{F}; M)$.

The term $E_2'^{p,q}$ is exactly $H_h^q H_v^p(A^{*,*}(S)^{\mathcal{F}})$. Using transfer as before we get that this equals $(H_h^q H_v^p(A^{*,*}(S)))^{\mathcal{F}}$ and by [10, p.352] we know that $H_h^q H_v^p(A^{*,*}(S)) = H^p(S/T; H^q(T; M))$. Hence the E_2 -page is as described in the statement of the Theorem as $(H_h^q H_v^p(A^{*,*}(S)))^{\mathcal{F}} = H^p(S/T; H^q(T; M))^{\mathcal{F}}$. \square

4. COMPARISON.

In this section we present some examples comparing our spectral sequence and Lyndon-Hochschild-Serre spectral sequence.

Let G be a finite group, $K \trianglelefteq G$ and $S \in \text{Syl}_p(G)$. Then $T = K \cap S$ is strongly \mathcal{F} -closed in $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{H} = \mathcal{F}_T(K)$ is normal in \mathcal{F} [2, 6.3]. By [2, 8.8] we have that $\mathcal{F}/\mathcal{H} = \mathcal{F}/T \cong \mathcal{F}_{SK/K}(G/K)$ with $SK/K \cong S/S \cap K = S/T$. Fix the $\mathbb{Z}_{(p)}$ -module M with trivial G -action. The Lyndon-Hochschild-Serre spectral sequence of the extension of groups is

$$H^p(G/K; H^q(K; M)) \Rightarrow H^{p+q}(G; M) = H^{p+q}(\mathcal{F}; M)$$

meanwhile the spectral sequence from Theorem 1.1 is

$$H^p(S/T; H^q(T; M))^{\mathcal{F}_S(G)} \Rightarrow H^{p+q}(G; M) = H^{p+q}(\mathcal{F}; M).$$

Using [6, XII.10.1] we can rewrite the E_2 -page of the Lyndon-Hochschild-Serre spectral sequence as $H^p(S/T; H^q(T; M))^{\mathcal{F}_T(K)}_{\mathcal{F}_{S/T}(G/K)}$. Both E_2 -pages coincide on the horizontal axis with values

$$H^p(G/K; M^K) = H^p(G/K; M) = H^p(S/T; M)^{\mathcal{F}_{S/T}(G/K)} = H^p(S/T; M)^{\mathcal{F}_S(G)}$$

because of Remark 3.1. On the vertical axis we have

$$H^q(K; M)^{G/K} = (H^q(T; M))^{\mathcal{F}_T(K)}_{G/K}$$

for the Lyndon-Hochschild-Serre spectral sequence and

$$(H^q(T; M)^{S/T})^{\mathcal{F}_S(G)}$$

for the spectral sequence of Theorem 1.1, where the action of $\mathcal{F}_S(G)$ on cohomology is defined in Lemma 3.2.

4.1. Remark. In the case $T = S$, i.e., with S being a Sylow p -subgroup both of K and G , and trivial coefficients $M = \mathbb{F}_p$, both spectral sequences collapse into the vertical axis of the E_2 -page with values $(H^q(S; M)^{\mathcal{F}_S(K)})^{G/K} = H^q(S; M)^{\mathcal{F}_S(G)}$. For the Lyndon-Hochschild-Serre spectral sequence the term $E_2^{p,q}$ vanishes for $p > 0$ as G/K is a p' -group. For the spectral sequence of Theorem 1.1 the same conclusion holds as $S/T = 1$.

Because of the earlier remarks the pages E_2 -pages may differ on the first quadrant except on the horizontal axis. We present two examples: in the first of them both E_2 -pages agree, on the second one one they do not.

4.2. Example. Fix $G = S_4$, the symmetric group on 4 letters, $K = A_4$, the alternating group on 4 letters, and $M = \mathbb{F}_2$, the field of two elements. Hence we have a short exact sequence $A_4 \rightarrow S_4 \rightarrow \mathbb{Z}_2$.

Write $x = (1234)$, $t = (12)(34)$, $z = x^2$. Then $S = \langle x, t \rangle \cong D_8$ is a Sylow 2-subgroup in S_4 , $T = A_4 \cap S = \langle z, t \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \trianglelefteq G$ is the Sylow 2-subgroup of A_4 and we have the short exact sequence of 2-groups:

$$T \rightarrow S \rightarrow \mathbb{Z}_2.$$

Consider the fusion system $\mathcal{F} = \mathcal{F}_S(G)$. The E_2 -page $H^p(\mathbb{Z}_2; H^q(T; \mathbb{F}_2))^{\mathcal{F}}$ is a subalgebra of the Lyndon-Hochschild-Serre E_2 -page $H^p(\mathbb{Z}_2; H^q(T; \mathbb{F}_2))$ associated to $T \rightarrow S \rightarrow \mathbb{Z}_2$. Write $S = \langle zt, t \rangle \rtimes \mathbb{Z}_2 = \mathbb{Z}_2 \wr \mathbb{Z}_2$ with \mathbb{Z}_2 exchanging the generators zt and t and set a and b to the duals of zt and t . Then $H^*(\mathbb{Z}_2 \times \mathbb{Z}_2)^{\mathbb{Z}_2} = \mathbb{F}_2[a, b]^{\mathbb{Z}_2} = \mathbb{F}_2[a + b, ab]$, $H^*(\mathbb{Z}_2; \mathbb{F}_2) = \mathbb{F}_2[y]$ and the corner of E_2 -page of the Lyndon-Hochschild-Serre spectral sequence of the dihedral group D_8 is

$$\begin{array}{ccc} \mathbb{F}_2\langle (a+b)^{\dots}, (a+b)ab \rangle & \bullet & \bullet \\ \mathbb{F}_2\langle (a+b)^2, ab \rangle & \bullet & \bullet \\ \mathbb{F}_2\langle a+b \rangle & \bullet & \bullet \\ \cdots \cdots \cdots \mathbb{F}_2 & \cdots \cdots \cdots \mathbb{F}_2\langle y \rangle & \cdots \cdots \cdots \mathbb{F}_2\langle y^2 \rangle \cdots \end{array}$$

with the dots representing some \mathbb{F}_p -vector space. All the differentials are zero and the spectral sequence collapses to give $H^*(D_8) = \mathbb{F}_2[y_1, \sigma_1, \sigma_2]/(y\sigma_1)$ [1, p.123] where the subindex denotes also degree and with $\sigma_1 = a + b$ and $\sigma_2 = ab$.

Next we compute the \mathcal{F} -stable elements in this E_2 -page. One checks that to be \mathcal{F} -stable is enough to be stable with respect to $\varphi = c_{(23)} : P = \langle t \rangle \rightarrow \langle z \rangle \leq S$. Notice that the E_2 -page of the Lyndon-Hochschild-Serre spectral sequence of the extension

$$0 \rightarrow P = P \cap T \rightarrow P \rightarrow 1 \rightarrow 0$$

is concentrated in the vertical axis as $H^*(\mathbb{Z}_2; \mathbb{F}_2) = \mathbb{F}_2[w]$, with w the dual of t . A short calculation yields

$$\varphi^*(a+b) = w, \varphi^*(ab) = 0$$

and

$$\text{res}(a+b) = 0, \text{res}(ab) = w^2.$$

Then it is immediate that the \mathcal{F} -stable elements in the corner of the E_2 -page are

$$\begin{array}{ccc} \mathbb{F}_2\langle (a+b)ab \rangle & \bullet & \bullet \\ \mathbb{F}_2\langle (a+b)^2 + ab \rangle & \bullet & \bullet \\ \vdots & \vdots & \vdots \\ \mathbb{F}_2 & \mathbb{F}_2\langle y \rangle & \mathbb{F}_2\langle y^2 \rangle \dots \end{array}$$

with zero differential. This gives $H^*(S_4; \mathbb{F}_2) = \mathbb{F}_2[y_1, \tau_2, \tau_3]/(y_1 \cdot \tau_3)$ where the index denote degree and with $\tau_2 = (a+b)^2 + ab$ and $\tau_3 = (a+b)ab$ [1, VI.1.13]. In this case the E_2 -page of stable elements coincide with the E_2 -page of the Lyndon-Hochschild-Serre spectral sequence of the extension $A_4 \rightarrow S_4 \rightarrow \mathbb{Z}_2$. To see why write $E_2^{p,q} = H^p(S_4/A_4; H^q(A_4; \mathbb{F}_2)) = H^p(\mathbb{Z}_2; H^q(T; \mathbb{F}_2)^{\mathcal{F}_T(A_4)} \mathcal{F}_{\mathbb{Z}_2}(\mathbb{Z}_2))$ for the E_2 -page of the latter spectral sequence. Notice that $T \trianglelefteq S_4$, $\mathcal{F}_{\mathbb{Z}_2}(\mathbb{Z}_2)$ is the trivial fusion system and $H^*(T; \mathbb{F}_2)^{A_4} = H^*(T; \mathbb{F}_2)^{S_4}$. Thus we can rewrite $E_2^{p,q}$ as $H^p(\mathbb{Z}_2; H^q(T; \mathbb{F}_2)^{S_4})$. This coincides with the term $H^p(\mathbb{Z}_2; H^q(T; \mathbb{F}_2))^{\mathcal{F}_S(S_4)} = H^p(\mathbb{Z}_2; H^q(T; \mathbb{F}_2)^{\mathcal{F}_T(S_4)})$.

4.3. Example. Consider $G = S \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ with $S = \mathbb{Z}_p \times \mathbb{Z}_p$ and p odd and where the first copy of \mathbb{Z}_2 swaps coordinates and the second copy of \mathbb{Z}_2 inverts both coordinates. Set K to the subgroup generated by the diagonal subgroup $T \cong \mathbb{Z}_p$ of S and the \mathbb{Z}_2 that swaps coordinates. Then $K \trianglelefteq G$, $T = S \cap K$ and $G/K \cong \mathbb{Z}_2$. We have the short exact sequence of p -groups

$$T \xrightarrow{\Delta} S = \mathbb{Z}_p \times \mathbb{Z}_p \rightarrow S/T \cong \mathbb{Z}_p.$$

We compute both $H^p(\mathbb{Z}_p; H^q(T; \mathbb{F}_p)^{\mathcal{F}_T(K)} \mathcal{F}_{\mathbb{Z}_p}(G/K))$ and $H^p(\mathbb{Z}_p; H^q(T; \mathbb{F}_p))^{\mathcal{F}_S(G)}$. Write $H^*(T; \mathbb{F}_p) = \Lambda(x) \otimes \mathbb{F}_p[y]$ with x in degree 1 and y in degree 2. As K acts trivially on T then $H^*(T; \mathbb{F}_p)^{\mathcal{F}_T(K)} = H^*(T; \mathbb{F}_p)$. So if we set V^q to the \mathbb{F}_p -vector space $H^q(T; \mathbb{F}_p)^{\mathcal{F}_T(K)}$ we have $V^q \cong \mathbb{F}_p$ for each $q \geq 0$. Because the action of $S/T \cong \mathbb{Z}_p$ on T is trivial we get

$$H^*(\mathbb{Z}_p; H^q(T; \mathbb{F}_p)^{\mathcal{F}_T(K)}) = \Lambda((V^q)^*) \otimes (V^q)^*,$$

where $(V^q)^*$ is the dual of V^q . The action of G/K on the quotient $S/T \cong \mathbb{Z}_p$ and on T becomes inversion and hence

$$H^p(\mathbb{Z}_p; H^q(T; \mathbb{F}_p)^{\mathcal{F}_T(K)} \mathcal{F}_{\mathbb{Z}_p}(G/K)) = \begin{cases} \mathbb{F}_p, & \text{for } p \text{ and } q \text{ equal to } 3, 4 \pmod{4}, \\ \mathbb{F}_p, & \text{for } p \text{ and } q \text{ equal to } 1, 2 \pmod{4}, \\ 0, & \text{otherwise.} \end{cases}$$

Now we compute $H^p(\mathbb{Z}_p; H^q(T; \mathbb{F}_p))^{\mathcal{F}_S(G)}$. Set W^q to the \mathbb{F}_p -vector space $H^q(T; \mathbb{F}_p)$. Hence $W^q \cong \mathbb{F}_p$ for each $q \geq 0$. Again because the action of $S/T \cong \mathbb{Z}_p$ on T is trivial we have

$$H^*(\mathbb{Z}_p; H^q(T; \mathbb{F}_p)) = \Lambda((W^q)^*) \otimes (W^q)^*,$$

where $(W^q)^*$ is the dual of W^q . The copy of \mathbb{Z}_2 of G that acts inverting coordinates on S inverts the generators of both T and $S/T \cong \mathbb{Z}_p$. The copy \mathbb{Z}_2 of G that swaps

$$H^p(\mathbb{Z}_p; H^q(T; \mathbb{F}_p))^{\mathcal{F}_S(G)} = \begin{cases} \mathbb{F}_p, & \text{for } p \text{ and } q \text{ equal to } 3, 4 \pmod{4}, \\ 0, & \text{otherwise.} \end{cases}$$

5. TATE'S THEOREM.

Associated to every first quadrant spectral sequence there is a five terms exact sequence. In the case of the Lyndon-Hochschild-Serre spectral sequence for $K \trianglelefteq G$ and the G -module M we obtain the inflation-restriction exact sequence:

where the second arrow from the right is the transgression. The five terms exact sequence for the spectral sequence of Theorem 1.1 for the fusion system \mathcal{F} over the p -subgroup S with strongly \mathcal{F} -closed subgroup T and $\mathbb{Z}_{(p)}$ -module M with trivial S -action is

where the arrow $H^1(T; M)^{\mathcal{F}} \rightarrow H^2(S/T; M)^{\mathcal{F}}$ is the transgression. Recall that $H^1(\mathcal{F}; M) = H^1(S; M)^{\mathcal{F}}$ and $H^2(\mathcal{F}; M) = H^2(S; M)^{\mathcal{F}}$. The inclusion of the E_2 -page of the spectral sequence of Theorem 1.1 into the E_2 -page of the Lyndon-Hochschild-Serre spectral sequence for $T \rightarrow S \rightarrow S/T$ induces a restriction map of five terms exact sequences:

Notice the following:

- (1) The maps g_1 and g_2 are injections as $H^1(S; M)^{\mathcal{F}}$ and $H^2(S; M)^{\mathcal{F}}$ are subgroups of $H^1(S; M)$ and $H^2(S; M)$ respectively.
(2) $H^1(S; \mathbb{F}_p) = \text{Hom}_{\mathbf{Ab}}(S, \mathbb{F}_p) = \text{Hom}_{\mathbf{Ab}}(S/S^p[S, S], \mathbb{F}_p)$ and

$$H^1(\mathcal{F}; \mathbb{F}_p) = \text{Hom}_{\mathbf{Ab}}(S/S^p[S, \mathcal{F}], \mathbb{F}_p)$$

where $[S, \mathcal{F}] = \langle [P, \text{Aut}_{\mathcal{F}}(P)] | P \leq S \rangle$ is the focal subgroup of S .

- (3) $H^1(T; \mathbb{F}_p)^{S/T} = \text{Hom}_{S/T}(T/T^p[T, T], \mathbb{F}_p) = \text{Hom}_{\mathbf{Ab}}(T/T^p[T, S], \mathbb{F}_p)$ and hence

$$(H^1(T; \mathbb{F}_p)^{S/T})^{\mathcal{F}} = \text{Hom}_{\mathbf{Ab}}(T/T^p[T, \mathcal{F}], \mathbb{F}_p).$$

- (4) $H^1(S/T; M)^{\mathcal{F}} = H^1(S/T; M)^{\mathcal{F}/T}$ and $H^2(S/T; M)^{\mathcal{F}} = H^2(S/T; M)^{\mathcal{F}/T}$ by Remark 3.1.

Now we are ready to prove Tate's theorem.

5.3. Theorem ([14]). *Let \mathcal{F} be a fusion system over the p -group S . If the restriction map $H^1(\mathcal{F}; \mathbb{F}_p) \rightarrow H^1(S; \mathbb{F}_p)$ is an isomorphism then $\mathcal{F} = \mathcal{F}_S(S)$.*

Proof. The isomorphism in the statement is equivalent to $S^p[S, S] = S^p[S, \mathcal{F}]$ by (2) above. Along the proof we will need to consider the hyperfocal subgroup of \mathcal{F}

$$O_{\mathcal{F}}^p(S) = \langle [P, O^p(\text{Aut}_{\mathcal{F}}(P))] | P \leq S \rangle.$$

Define a series of subgroups of S by $S_0 = S$ and $S_{n+1} = S_n^p[S_n, S]$. Define another series of subgroups of S by $T_0 = S$ and $T_{n+1} = T_n^p[T_n, \mathcal{F}]$. The hypothesis reads now as $S_1 = T_1$. Moreover, T_1 is strongly \mathcal{F} -closed and contains $O_{\mathcal{F}}^p(S)$ by [9, A.6]. We proof by induction that this is the case for any $n \geq 1$, i.e., that T_n is strongly \mathcal{F} -closed, contains $O_{\mathcal{F}}^p(S)$ and $S_n = T_n$ for any $n \geq 1$.

So assume the hypothesis holds for T_n . As T_n is strongly \mathcal{F} -closed in S we have a restriction map as above for $M = \mathbb{F}_p$ coefficients:

$$\begin{array}{ccccccccc} 0 & \rightarrow & H^1(S/T_n)^{\mathcal{F}} & \rightarrow & H^1(\mathcal{F}) & \rightarrow & (H^1(T_n)^{S/T_n})^{\mathcal{F}} & \rightarrow & H^2(S/T_n)^{\mathcal{F}} & \rightarrow & H^2(\mathcal{F}) \\ & & \downarrow f_1 & & \downarrow g_1 & & \downarrow h_1 & & \downarrow f_2 & & \downarrow g_2 \\ 0 & \rightarrow & H^1(S/T_n) & \rightarrow & H^1(S) & \rightarrow & H^1(T_n)^{S/T_n} & \rightarrow & H^2(S/T_n) & \rightarrow & H^2(S). \end{array}$$

Because $O_{\mathcal{F}}^p(S) \leq T_n$ the quotient \mathcal{F}/T_n is a p -group, i.e., $\mathcal{F}/T_n = \mathcal{F}_{S/T_n}(S/T_n)$. Hence by point (4) before the proof we have that both f_1 and f_2 are isomorphisms. Also by hypothesis we have that $T_1 = S_1$ and hence g_1 is an isomorphisms by (2) above. Hence by the five lemma and (1) we get that h_1 is an isomorphism. Then by (3) we deduce that $T_n^p[T_n, S] = T_n^p[T_n, \mathcal{F}]$, i.e., that $S_{n+1} = T_{n+1}$. To prove that $O_{\mathcal{F}}^p(S)$ is contained in T_{n+1} consider the unique p -power index fusion subsystem \mathcal{F}_{T_n} of \mathcal{F} on T_n [4, 4.3]. Then by [9, A.14] we have that $O_{\mathcal{F}_{T_n}}^p(T_n) = O_{\mathcal{F}}^p(S)$. By [9, A.6] we have that $O_{\mathcal{F}}^p(S) \leq T_n^p[T_n, \mathcal{F}_{T_n}] \leq T_n^p[T_n, \mathcal{F}] = T_{n+1}$. Finally, as S_{n+1} is normal in S and so it is T_{n+1} . Then by [9, A.7(1)] T_{n+1} is also strongly \mathcal{F} -closed in \mathcal{F} . This finishes the induction.

We have proven in particular that $O_{\mathcal{F}}^p(S) \leq T_n = S_n$ for each $n \geq 1$. It is clear that S_n is the trivial group for n big enough as S is a finite p -group. Hence we deduce that $O_{\mathcal{F}}^p(S) = 1$ and this implies that there are no p' -automorphisms in \mathcal{F} , i.e., $\mathcal{F} = \mathcal{F}_S(S)$. \square

REFERENCES

- [1] A. Adem and R.J. Milgram, *Cohomology of Finite Groups*, SpringerVerlag Grundlehren 309 (2004).
- [2] Aschbacher M., *Normal subsystems of fusion systems*, PLMS (3) 97 (2008), pp. 239-271.
- [3] C. Broto, R. Levi and B. Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. 16 (2003), 779-856.
- [4] C. Broto, N. Castellana, J. Grodal, R. Levi and B. Oliver *Extensions of p -local finite groups*, Trans. Amer. Math. Soc. 359 (2007), 3791-3858.
- [5] J. Cantarero, J. Scherer and A. Viruel, *Nilpotent p -local finite groups*, preprint 2011, arXiv:1107.5158v1 [math.AT]
- [6] H. Cartan and S. Eilenberg, *Homological Algebra*, Princenton University Press 1956.
- [7] D. Craven, *Normal subsystems of fusion systems*, J. London Math. Soc. (2011), doi: 10.1112/jlms/jdr004
- [8] D. Craven, *Control of Fusion and Solubility in Fusion Systems*, J. Algebra 323 (2010), no. 9, 2429–2448.
- [9] A. Daz, A. Glessner, R. Stancu and S. Park, *Tate’s and Yoshida’s theorems on control of transfer for fusion systems*, J. London Math. Soc. (2011), to appear, arXiv:1002.4343v1 [math.GR]
- [10] S. Mac Lane, *Homology*, Springer-Verlag 1963.
- [11] Park S. *Realizing a fusion system by a single finite group*, Archiv der Mathematik, Volume 94, Number 5, 405-410, DOI: 10.1007/s00013-010-0119-z
- [12] Ragnarsson, Stancu, *Saturated fusion systems as idempotents in the double burnside ring*, preprint (2010), arXiv:0911.0085v3 [math.AT]
- [13] Solomon R., Stancu R., *Conjectures on finite and p -local groups*, L’Enseignement Mathématique 54, (2008), 61–66.
- [14] J. Tate, *Nilpotent quotient groups*, Topology 3, (1964) suppl. 1, 109-111.
- [15] Thevenaz, Webb, *The structure of Mackey functors*, Trans. Amer. Math. Soc. 347, No. 6 (Jun., 1995), pp. 1865-1961.

DEPARTAMENTO DE ÁLGEBRA, GEOMETRÍA Y TOPOLOGÍA, UNIVERSIDAD DE MÁLAGA, APDO CORREOS 59, 29080 MÁLAGA, SPAIN.

E-mail address: `adiaz@agt.cie.uma.es`